

ON THE TORSION UNITS OF THE INTEGRAL GROUP RING OF FINITE PROJECTIVE SPECIAL LINEAR GROUPS

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ABSTRACT. H. J. Zassenhaus conjectured that any unit of finite order and augmentation one in the integral group ring of a finite group G is conjugate in the rational group algebra to an element of G . One way to verify this is showing that such unit has the same distribution of partial augmentations as an element of G and the HeLP Method provides a tool to do that in some cases. In this paper we use the HeLP Method to describe the partial augmentations of a hypothetical counterexample to the conjecture for the projective special linear groups.

1. INTRODUCTION

Let G be a finite group and let $\mathbb{Z}G$ be the integral group ring of G . Denote by $V(\mathbb{Z}G)$ the group of normalized units (i.e. units of augmentation 1) in $\mathbb{Z}G$. Hans Zassenhaus stated a list of conjectures about finite subgroups of $V(\mathbb{Z}G)$. In this paper we deal with the unique one which is still open. It states that every torsion element of $V(\mathbb{Z}G)$ is rationally conjugate (i.e. conjugate in the units of $\mathbb{Q}G$) to an element of G . We refer to this statement as the Zassenhaus Conjecture. It has been verified for nilpotent groups [Wei91], for cyclic-by-abelian groups [CMdR13] and for p -group-by-abelian p' -groups [Her06]. For non-solvable groups the Zassenhaus Conjecture has been proved only for a few simple or almost simple groups [HP72, LP89, LT91, Her08, Gil13, KK13, Her07, BM14, Mar15, BM, MdRS16]. Actually all the simple groups for which the Zassenhaus Conjecture has been proved are of the form $\text{PSL}(2, p^f)$, i.e. projective special linear groups.

Suppose that $G = \text{PSL}(2, p^f)$ with p a prime integer and let u be an element of order n in $V(\mathbb{Z}G)$. Hertweck proved that u is rationally conjugate to an element of G provided that n is prime different from p , or $n = p$ and $f \leq 2$, or $n = 6$ [Her07]. This was extended by Margolis to p -regular elements of prime power order [Mar16]. Recently Margolis, del Río and Serrano have proved that u is rationally conjugate to an element of G if n is coprime with $2p$ [MdRS16]. The aim of this paper is showing that the main tool used to prove these results fails for the next natural case to consider, namely when $n = 2t$ with t prime and greater than 4. On the positive side the main result of the paper provides significant information on a possible counterexample to the Zassenhaus Conjecture of this kind.

The tool mentioned in the previous paragraph is the HeLP Method which was introduced by Luthar and Passi in [LP89] and improved by Hertweck in [Her07]. The basic idea of the HeLP Method, for G an arbitrary finite group, is as follows: Given an element $u \in V(\mathbb{Z}G)$ of order n , we call *distribution of partial augmentations* of u to the partial augmentations of the elements u^d with d running on the divisors of n . Let

$$\text{TPA}_n(G) = \{\text{Distributions of partial augmentations of elements of } G \text{ of order } n\}$$

and

$$\text{PA}_n(G) = \{\text{Distributions of partial augmentations of elements of } V(\mathbb{Z}G) \text{ of order } n\}.$$

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By a theorem of Marciniak, Ritter, Sehgal and Weiss, every element of $V(\mathbb{Z}G)$ of order n is rationally conjugate to an element of G if and only if $\text{TPA}_n(G) = \text{PA}_n(G)$ (see Theorem 2.1.(4)). Calculating $\text{TPA}_n(G)$ is very easy but, unfortunately, calculating $\text{PA}_n(G)$ is usually difficult. The HeLP Method consists in calculating a set $\text{VPA}_n(G)$ containing $\text{PA}_n(G)$ which will be defined in Section 2. We call the elements of $\text{VPA}_n(G)$, *distributions of virtual partial augmentations* of order n for G , because they satisfy some properties known for distributions of partial augmentations of elements of order n in $V(\mathbb{Z}G)$. If $\text{VPA}_n(G) = \text{TPA}_n(G)$ then all elements of $V(\mathbb{Z}G)$ of order n are rationally conjugate to elements of G and, in case this holds for all the possible orders n , then the Zassenhaus Conjecture holds for G . The HeLP Method fails to verify the Zassenhaus Conjecture when $\text{VPA}_n(G) \neq \text{TPA}_n(G)$ for some n . Nevertheless, each element of $\text{VPA}_n(G) \setminus \text{TPA}_n(G)$ provides relevant information of a possible counterexample to the Zassenhaus Conjecture. Indeed, it determines a conjugacy class in the normalized units of $\mathbb{Q}G$ formed by elements with integral partial augmentations. Actually, using the representation theory of $\mathbb{Q}G$ one can find a concrete representative of this class. With this information at hand, to prove the Zassenhaus Conjecture one should prove that none of these conjugacy classes contains an element with integral coefficients and to disprove it one should find an element in this class with integral coefficients. In this paper we prove that for $G = \text{PSL}(2, p^f)$ and t an odd prime different from p the elements of $\text{VPA}_{2t}(G) \setminus \text{TPA}_{2t}(G)$ are of a very specific form (see Theorem 2.3). So, it could be potentially used to prove or disprove the Zassenhaus Conjecture in this case, but completing this would require new techniques.

The paper is organized as follows: In Section 2 we introduce the basic notation of the paper, explain the HeLP Method and state the main result of the paper. In Section 3 we collect some general technical results. In Section 4 we recall the representation theory of a projective special linear group $G = \text{PSL}(2, q)$ and calculate the sums $\tilde{\Upsilon}_d(m) = \sum_{g \in G, |g|=m} \Upsilon_d(g)$ for $\Upsilon \in \text{VPA}_{rt}(G)$ for r and t two different primes not dividing q and $d \mid rt$. We call these sums the *accumulated virtual partial augmentations*. Finally, in Section 5 we prove Theorem 2.3.

2. THE HELP METHOD AND THE MAIN RESULT

In this section we introduce the basic notation and state the main result of the paper.

Let $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers. For a positive integer n we always use ζ_n to denote a complex primitive n -th root of unity. If F/K is an extension of number fields then $\text{Tr}_{F/K} : F \rightarrow K$ denotes the trace map.

Let G be a finite group and let $g \in G$. Then $|g|$ denotes the order of g , $\langle g \rangle$ denotes the cyclic group generated by g and g^G denotes the conjugacy class of g in G . If α is an element of a group ring of G and α_g denotes the coefficient of an element g of G , then the *partial augmentation* of α at g is

$$\varepsilon_g(\alpha) = \sum_{h \in g^G} \alpha_h.$$

The following theorems collect some well known facts, the first one on partial augmentations of torsion elements in $V(\mathbb{Z}G)$ and the second one on torsion elements of $V(\mathbb{Z} \text{PSL}(2, q))$. See [Seh93, 1.5, 7.3, 41.5] or [Ber55, Hig40, MRSW87] and [Her07, Mar16].

Theorem 2.1. *Let G be a finite group and let u be an element of order n in $V(\mathbb{Z}G)$. Then the following statements hold:*

- (1) *If $u \neq 1$ then $\varepsilon_1(u) = 0$ (Berman-Higman Theorem).*
- (2) *n divides the exponent of G .*
- (3) *If $g \in G$ and $\varepsilon_g(u) \neq 0$ then the order of g divides n .*
- (4) *The following statements are equivalent.*
 - (a) *u is rationally conjugate to an element of G .*
 - (b) *For every $d \mid n$ there exists an element $g_0 \in G$ such that $\varepsilon_g(u^d) = 0$ for every $g \in G \setminus g_0^G$.*
 - (c) *$\varepsilon_g(u^d) \geq 0$ for every $g \in G$ and every $d \mid n$.*

Theorem 2.2. *Let $G = \text{PSL}(2, q)$ for an odd prime power q , and let u be an element in $V(\mathbb{Z}G)$ of order n . Then the following statements hold:*

- (1) *If either $\gcd(n, q) = 1$ or q is prime then G has an element of order n .*
- (2) *If n is a prime power and $\gcd(n, q) = 1$ then u is rationally conjugate to an element of G .*

If u is a unit of $\mathbb{C}G$ of order n then the *distribution of partial augmentations* of u is the list $(\Upsilon_d)_{d|n}$ of class functions of G given by $\Upsilon_d(g) = \varepsilon_g(u^d)$, for every $g \in G$ and every divisor d of n . It is easy to see that every conjugate to u in $\mathbb{C}G$ has the same distribution of partial augmentations as u . In case u and u' are elements of $V(\mathbb{Z}G)$ of order n with the same distribution of partial augmentations then u and u' are rationally conjugate. To explain this we need to introduce some notation. Let χ be an ordinary character of G . The linear span of χ to $\mathbb{C}G$ takes the form

$$(1) \quad \chi(u) = \sum_{x \in G} \varepsilon_x(u) \chi(x), \quad (u \in \mathbb{C}G)$$

where $\sum_{x \in G}$ is an abbreviation of $\sum_{x \in T}$ for T a set of representatives of the conjugacy classes of G . Let ρ be a representation of G affording χ . If $z \in \mathbb{C}$ then the multiplicity of z as an eigenvalue of $\rho(u)$ only depends on the character χ and it is denoted by $\mu(z, u, \chi)$. As $u^n = 1$, every eigenvalue of $\rho(u)$ is of the form ζ_n^l for some integer $0 \leq l \leq n-1$, and the following formula gives the multiplicity of ζ_n^l as an eigenvalue of $\rho(u)$ in terms of the partial augmentations (see [LP89]):

$$(2) \quad \mu(\zeta_n^l, u, \chi) = \frac{1}{n} \sum_{x \in G} \sum_{d|n} \varepsilon_x(u^d) \text{Tr}_{\mathbb{Q}(\zeta_n^d)/\mathbb{Q}}(\chi(x) \zeta_n^{-ld}).$$

Observe that the right side of the previous formula makes sense because if $\varepsilon_x(u^d) \neq 0$ then $x^{\frac{n}{d}} = 1$, by Theorem 2.1.(3), and hence $\chi(x) \in \mathbb{Q}(\zeta_n^d)$. Since u and u' have the same distribution of partial augmentations, Formula (2) implies that the images of u and u' by all the representations of G have the same eigenvalues with the same multiplicities. Thus u and u' are conjugate in $\mathbb{C}G$ and hence they are rationally conjugate (see [Seh93, Lemma 37.5]), as desired. This serves as a proof of Theorem 2.1.(4) and also explains why proving the Zassenhaus Conjecture for G is equivalent to prove that $\text{PA}_n(G) = \text{TPA}_n(G)$ for every n . Clearly,

$$\text{TPA}_n(G) = \left\{ (\Upsilon_d)_{d|n} : \text{there exists } g \in G \text{ with } \Upsilon_d(h) = 1 \text{ if } h \in (g^d)^G \text{ and } \Upsilon_d(h) = 0 \text{ otherwise} \right\},$$

but $\text{PA}_n(G)$ is usually hard to calculate.

The equality in (2) extends to Brauer characters modulo a prime p not dividing n . More precisely, let ρ be a representation of G in characteristic p and let χ be the Brauer character afforded by ρ with respect to a sufficiently large p -modular system. Let u be an element of order n in $V(\mathbb{Z}G)$ with n coprime with p and let $\mu(\zeta_n^l, u, \chi)$ denote the multiplicity of $\overline{\zeta_n^l}$ as an eigenvalue of $\rho(\overline{u})$, where the bar notation stands for reduction modulo p . This multiplicity can be calculated using (2) (see [Her07]). Again the formula makes sense because if $\varepsilon_g(u) \neq 0$ then g is p -regular, by Theorem 2.1.(3).

Formula (2) is the bulk of the HeLP Method. Namely, if $u \in V(\mathbb{Z}G)$ satisfies $u^n = 1$ and χ is either an ordinary character of G or a Brauer character of G modulo a prime not dividing n , then for every integer l

$$\frac{1}{n} \sum_{x \in G} \sum_{d|n} \varepsilon_x(u^d) \text{Tr}_{\mathbb{Q}(\zeta_n^d)/\mathbb{Q}}(\chi(x) \zeta_n^{-ld}) \in \mathbb{Z}_{\geq 0}.$$

Let n be a positive integer. A *distribution of virtual partial augmentations* of order n for G is a list $\Upsilon = (\Upsilon_d)_{d|n}$, indexed by the divisors of n , where each Υ_d is a class function of G taking values on \mathbb{Z} , and the following conditions hold:

- (V1) $\sum_{x \in G} \Upsilon_d(x) = 1$;
- (V2) if $d \neq n$ then $\Upsilon_d(1) = 0$;
- (V3) if $\frac{n}{d}$ is not multiple of $|x|$ then $\Upsilon_d(x) = 0$;

(V4) if χ is either an ordinary character of G or a Brauer character of G modulo a prime not dividing n and $l \in \mathbb{Z}$ then

$$\mu(\zeta_n^l, \Upsilon, \chi) = \frac{1}{n} \sum_{x \in G} \sum_{d|n} \Upsilon_d(x) \operatorname{Tr}_{\mathbb{Q}(\zeta_n^d)/\mathbb{Q}}(\chi(x) \zeta_n^{-ld}) \in \mathbb{Z}_{\geq 0}.$$

As for (2), the right side of the previous formula makes sense because, by (V3), if $\Upsilon_d(x) \neq 0$ then the order of x divides $\frac{n}{d}$ and hence x is p -regular and $\chi(x) \in \mathbb{Q}(\zeta_n^d)$. Let

$$\operatorname{VPA}_n(G) = \{\text{Distributions of virtual partial augmentations of order } n \text{ for } G\}.$$

We have

$$\operatorname{TPA}_n(G) \subseteq \operatorname{PA}_n(G) \subseteq \operatorname{VPA}_n(G).$$

Indeed, the first inclusion is obvious and the second one follows from the Bergman-Higman Theorem, Theorem 2.1.(3) and formula (2).

Let $G = \operatorname{PSL}(2, q)$ with q an odd prime power and let t be an odd prime. By Theorem 2.2.(1), $V(\mathbb{Z}G)$ has elements of order $2t$ if and only if so does G if and only if $q \equiv \pm 1 \pmod{4t}$. Thus we assume that $q \equiv \pm 1 \pmod{4t}$. For $g_0 \in G$ with $|g_0| = 2t$ and $t \geq 5$ let $\Upsilon^{(g_0)}$ denote the list of class functions $(\Upsilon_1^{(g_0)}, \Upsilon_2^{(g_0)}, \Upsilon_t^{(g_0)}, \Upsilon_{2t}^{(g_0)})$ of G defined as follows:

$$(3) \quad \Upsilon_d^{(g_0)}(g) = \begin{cases} 1, & \text{if } (d, g^G) \in \left\{ (2t, 1^G), (t, (g_0^t)^G), (2, (g_0^2)^G), (1, (g_0^{\frac{t-1}{2}})^G), (1, (g_0^{\frac{t+1}{2}})^G) \right\}; \\ -1, & \text{if } (d, g^G) = (1, (g_0^{t-1})^G); \\ 0, & \text{otherwise.} \end{cases}$$

As explained in the introduction we calculate $\operatorname{VPA}_{2t}(G)$, namely we prove the following.

Theorem 2.3. *Let t be an odd prime and let q be a prime power such that $q \equiv \pm 1 \pmod{4t}$ and let $G = \operatorname{PSL}(2, q)$. Then $\operatorname{VPA}_6(G) = \operatorname{TPA}_6(G)$ and if $t \geq 5$ then*

$$\operatorname{VPA}_{2t}(G) = \operatorname{TPA}_{2t}(G) \cup \{\Upsilon^{(g_0)} : g_0 \in G, |g_0| = 2t\}.$$

Suppose that $t \geq 5$ and let g_0 be an element of order $2t$ in $G = \operatorname{PSL}(2, q)$. Then $\Upsilon^{(g_0)}$ is the distribution of partial augmentations of the elements of a conjugacy class C in the units of $\mathbb{Q}G$ of an element of order $2t$ in $V(\mathbb{Q}G)$ with integral partial augmentations. To settle the Zassenhaus Conjecture in this case it remains to decide whether C contains an element u with integral coefficients. If not, the Zassenhaus Conjecture holds in this case and otherwise u provides a counterexample for the Zassenhaus Conjecture. The smallest example of this situation is encountered for $q = 19$ and $t = 5$. However, Bächle and Margolis has proved the Zassenhaus Conjecture for this example with a technique which they called the Lattice Method [BM14]. Unfortunately, the Lattice Method does not apply for the next cases ($q = 27$ and $q = 29$ and $t = 7$) basically because the representation type appearing in these cases is wild.

3. PRELIMINARY GENERAL RESULTS

In this section we collect some technical results that will be used in subsequent sections. We start quoting:

Lemma 3.1. [Mar16, Lemma 2.1] *If n and d are positive integers with $d \mid n$ then $\operatorname{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_d) = \mu(d) \frac{\varphi(n)}{\varphi(d)}$, where φ denotes the Euler's totient function and μ denotes the Möbius function.*

The following well known formula, for k and d integers with $k > 0$, will be used in several situations:

$$(4) \quad \sum_{i=0}^{k-1} \zeta_k^{-id} = \begin{cases} 0, & \text{if } k \nmid d; \\ k, & \text{otherwise.} \end{cases}$$

In the remainder of the section G is a finite group, n is a positive integer and $\Upsilon \in \text{VPA}_n(G)$. If m divides n then we define

$$\Upsilon^{\frac{n}{m}} = ((\Upsilon^{\frac{n}{m}})_d)_{d|m} \quad \text{with} \quad (\Upsilon^{\frac{n}{m}})_d(g) = \Upsilon_{d\frac{n}{m}}(g) \quad \text{for} \quad g \in G.$$

Observe that if Υ is the distribution of partial augmentations of an element $u \in V(\mathbb{Z}G)$ of order n then $\Upsilon^{\frac{n}{m}}$ is the distribution of partial augmentations of $u^{\frac{n}{m}}$. Moreover,

$$(5) \quad \text{if } \Upsilon \in \text{VPA}_n(G) \text{ and } m \mid n \text{ then } \Upsilon^{\frac{n}{m}} \in \text{VPA}_m(G).$$

Indeed, that $\Upsilon^{\frac{n}{m}}$ satisfies (V1), (V2) and (V3) is elementary and (V4) follows from the following lemma.

Lemma 3.2. *Let G be a finite group. Let n and m be positive integers with $m \mid n$, let $l \in \mathbb{Z}$ and let $\Upsilon \in \text{VPA}_n(G)$. Let χ be either an ordinary character of G or a Brauer character of G module a prime not dividing n . Then*

$$\mu(\zeta_m^l, \Upsilon^{\frac{n}{m}}, \chi) = \sum_{\xi, \xi^{\frac{n}{m}} = \zeta_m^l} \mu(\xi, \Upsilon, \chi).$$

Proof. Let $k = \frac{n}{m}$ and fix $\xi_0 \in \mathbb{C}$ with $\xi_0^k = \zeta_m^l$. Then $\xi^k = \zeta_m^l$ if and only if $(\xi\xi_0^{-1})^k = 1$ if and only if $\xi = \xi_0\zeta_k^i$ for some $i \in \{0, 1, \dots, k-1\}$. Then

$$\begin{aligned} \sum_{\xi, \xi^k = \zeta_m^l} \mu(\xi, \Upsilon, \chi) &= \frac{1}{n} \sum_{x \in G} \sum_{d|n} \Upsilon_d(x) \text{Tr}_{\mathbb{Q}(\zeta_n^d)/\mathbb{Q}} \left(\chi(x) \xi_0^{-d} \sum_{i=0}^{k-1} \zeta_k^{-id} \right) \\ &= \frac{1}{m} \sum_{x \in G} \sum_{d|n, k|d} \Upsilon_d(x) \text{Tr}_{\mathbb{Q}(\zeta_n^d)/\mathbb{Q}} (\chi(x) \xi_0^{-d}), \end{aligned}$$

where the last equality is a consequence of (4). Furthermore $\{d : d \mid n, k \mid d\} = \{kd_1 : d_1 \mid m\}$ and ζ_n^k has order m . Thus

$$\sum_{\xi, \xi^k = \zeta_m^l} \mu(\xi, \Upsilon, \chi) = \frac{1}{m} \sum_{x \in G} \sum_{d_1|m} (\Upsilon^{\frac{n}{m}})_{d_1}(x) \text{Tr}_{\mathbb{Q}(\zeta_m^{d_1})/\mathbb{Q}} (\chi(x) \zeta_m^{-ld_1}) = \mu(\zeta_m^l, \Upsilon^{\frac{n}{m}}, \chi).$$

□

Proposition 3.3. *Let G be a finite group and let n be a positive integer. If $\text{VPA}_n(G) \neq \emptyset$ then every prime divisor of n divides $|G|$.*

Proof. Let p be a prime not dividing $|G|$ and let $\Upsilon \in \text{VPA}_p(G)$. By (V2) and (V3), $\Upsilon_1(g) = 0$ for every $g \in G$ and this is in contradiction with (V1). This shows that if p does not divide the order of G then $\text{VPA}_p(G) = \emptyset$. Now, if $\Upsilon \in \text{VPA}_n(G)$, with p a prime divisor of n then $\Upsilon^{\frac{n}{p}} \in \text{VPA}_p(G)$ by (5), and hence p divides the order of G , by the previous sentence. □

Let χ be either an ordinary character of G or a Brauer character of G module a prime p not dividing n and let m be a divisor of n . Let

$$\mu_m^-(\Upsilon, \chi) = \frac{1}{n} \sum_{x \in G} \sum_{d|n, m \nmid d} \Upsilon_d(x) \text{Tr}_{\mathbb{Q}(\zeta_n^d)/\mathbb{Q}} (\chi(x)).$$

By (V4), for every $l \in \mathbb{Z}$, we have

$$0 \leq \mu(\zeta_m^l, \Upsilon, \chi) = \mu_m^-(\Upsilon, \chi) + \frac{1}{n} \sum_{x \in G} \sum_{d|n, m \nmid d} \Upsilon_d(x) \text{Tr}_{\mathbb{Q}(\zeta_n^d)/\mathbb{Q}} (\chi(x) \zeta_m^{-dl}).$$

Combining this with (4) we obtain

$$\begin{aligned}
0 &\leq \mu(1, \Upsilon, \chi) = \mu_m^-(\Upsilon, \chi) + \frac{1}{n} \sum_{x \in G} \sum_{d \mid n, m \nmid d} \Upsilon_d(x) \operatorname{Tr}_{\mathbb{Q}(\zeta_n^d)/\mathbb{Q}}(\chi(x)) \\
&= \mu_m^-(\Upsilon, \chi) - \frac{1}{n} \sum_{l=1}^{m-1} \sum_{x \in G} \sum_{d \mid n, m \nmid d} \Upsilon_d(x) \operatorname{Tr}_{\mathbb{Q}(\zeta_n^d)/\mathbb{Q}}(\chi(x) \zeta_m^{-dl}) \leq m \mu_m^-(\Upsilon, \chi).
\end{aligned}$$

We record this for future use:

$$(6) \quad 0 \leq \mu(1, \Upsilon, \chi) \leq m \mu_m^-(\Upsilon, \chi).$$

4. PRELIMINARY RESULTS FOR $\operatorname{PSL}(2, q)$

In this section p is an odd prime, $q = p^f$ with $f \geq 1$ and $G = \operatorname{PSL}(2, q)$. In the first part of the section we describe the ordinary and Brauer characters of G which will be used in the remainder of the paper. In the second part we first describe $\operatorname{VPA}_r(G)$ with r a prime different from p and then we calculate the accumulated virtual partial augmentations of elements in $\operatorname{VPA}_{rt}(G)$ with r and t different primes such that $q \equiv \pm 1 \pmod{2rt}$.

Suppose that G has an element g_0 of order m with $p \nmid m$. Then $q \equiv \epsilon \pmod{2m}$ with $\epsilon = \pm 1$. Moreover, every element of G of order dividing m is conjugate to some element of $\langle g_0 \rangle$ and g_0^i and g_0^j are conjugate in G if and only if $i \equiv \pm j \pmod{m}$.

If h is an integer then let $\phi_h, \psi_h : \langle g_0 \rangle \rightarrow \mathbb{C}$ be defined as follows:

$$\phi_h(g_0^i) = \begin{cases} q + \epsilon, & \text{if } m \mid i; \\ \epsilon(\zeta_m^{hi} + \zeta_m^{-hi}), & \text{otherwise;} \end{cases} \quad \psi_h(g_0^i) = \begin{cases} q - \epsilon, & \text{if } m \mid i; \\ 0, & \text{otherwise.} \end{cases}$$

Given $R = (r_0, r_1, \dots, r_k) \in \mathbb{Z}^{k+1}$, let

$$X_R = \{(s_0, s_1, \dots, s_k) \in \mathbb{Z}^{k+1} : -r_j \leq s_j \leq r_j \text{ and } 2 \mid r_j - s_j \text{ for each } j\}.$$

If moreover $r_0 + \dots + r_k$ is even then let

$$(7) \quad V_{R;h} = \left\{ (s_0, s_1, \dots, s_k) \in X_R \setminus \{(0, \dots, 0)\} : \frac{\sum_{j=0}^k s_j p^j}{2} \equiv \pm h \pmod{m} \right\}$$

and let $\chi_{r_0, r_1, \dots, r_k} : \langle g_0 \rangle \rightarrow \mathbb{C}$ be defined by

$$\chi_{r_0, r_1, \dots, r_k}(g_0^i) = \sum_{(s_0, s_1, \dots, s_k) \in X_R} \zeta_m^{i \frac{\sum_{j=0}^k s_j p^j}{2}}.$$

Lemma 4.1. *Let g_0 be a p -regular element of G of order m . Then the following statements hold for $s \in \mathbb{Z}$ and $R = (r_0, r_1, \dots, r_k) \in \mathbb{Z}^{k+1}$ with $r_0 + r_1 + \dots + r_k$ even:*

- (1) *If $m \nmid s$ then both ϕ_s and ψ_s are the restriction to $\langle g_0 \rangle$ of ordinary characters of G .*
- (2) *$\chi_{r_0, r_1, \dots, r_k}$ is the restriction to $\langle g_0 \rangle$ of a Brauer character of G modulo p .*
- (3) *If χ is an irreducible Brauer character of G modulo p then the restriction of χ to $\langle g_0 \rangle$ equals $\chi_{r_0, r_1, \dots, r_{f-1}}$ for some integers $0 \leq r_0, r_1, \dots, r_{f-1} \leq p-1$ with $r_0 + r_1 + \dots + r_{f-1}$ even.*
- (4)

$$\chi_{r_0, r_1, \dots, r_k} = \begin{cases} (1 + 2n_0)1_G + \epsilon \sum_{h=1}^{\frac{m}{2}} n_h (\phi_h - \psi_h), & \text{if } 2 \mid r_j \text{ for all } j; \\ 2n_0 1_G + \epsilon \sum_{h=1}^{\frac{m}{2}} n_h (\phi_h - \psi_h), & \text{otherwise;} \end{cases}$$

where $2n_h = |V_{R;h}|$ and 1_G denotes the trivial character of G .

Proof. (1) To prove that ϕ_s and ψ_s are the restriction to $\langle g_0 \rangle$ of ordinary characters of G we simply express them in terms of the irreducible characters η_1, η_2, θ_i and χ_i of G as described in Table 2 of [Her07]. If $s \equiv \pm s' \pmod m$ then $\phi_s = \phi_{s'}$ and $\psi_s = \psi_{s'}$. Therefore we may assume that $1 \leq s \leq \frac{m}{2}$. Firstly, if m is even then $\phi_{\frac{m}{2}}$ is the restriction of $\eta_1 + \eta_2$, and $\psi_{\frac{m}{2}}$ is the restriction of any θ_j if $\epsilon = 1$, and the restriction of any χ_i if $\epsilon = -1$. This covers the case $s = \frac{m}{2}$. Suppose otherwise that $1 \leq s < \frac{m}{2}$. If $\epsilon = 1$ then ϕ_s is the restriction of $\chi_{s\frac{q-1}{2m}}$ and ψ_s is the restriction of $\theta_{s\frac{q-1}{2m}}$, while if $\epsilon = -1$ then ϕ_s is the restriction of $\theta_{s\frac{q+1}{2m}}$ and ψ_s is the restriction of $\chi_{s\frac{q+1}{2m}}$.

(2) Let K be a field of characteristic p . The following defines an action by K -automorphisms on the group $\text{SL}(2, q)$ on the ring of polynomials $K[X, Y]$ [Alp86, Pages 14–16]:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X = aX + bY, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} Y = cX + dY.$$

If n is a positive integer then the vector space V_n formed by the homogenous polynomials of degree n is invariant under this action and, if moreover n is even then $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on V_n . Therefore G acts on V_n provided that n is even.

Let us fix integers $0 \leq r_0, r_1, \dots, r_k \leq p-1$ such that $r_0 + r_1 + \dots + r_k$ is even and let $n = r_0 + r_1p + \dots + r_kp^k$. Then n is even. Let W_{r_0, r_1, \dots, r_k} be the subspace of V_n generated by the polynomials of the form $X^i Y^{n-i}$ with $i = i_0 + i_1p + \dots + i_kp^k$ and $0 \leq i_j \leq r_j$ for every j . For such s we have

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} X^i Y^{n-i} &= (aX + bY)^i (cX + dY)^{n-i} \\ &= \prod_{h=0}^k \left(a^{p^h} X^{p^h} + b^{p^h} Y^{p^h} \right)^{i_h} \left(c^{p^h} X^{p^h} + d^{p^h} Y^{p^h} \right)^{r_h - i_h} \\ &= \prod_{h=0}^k \left(\sum_{u=0}^{i_h} \alpha_{h,u} X^{up^h} Y^{(i_h-u)p^h} \right) \left(\sum_{v=0}^{r_h - i_h} \beta_{h,v} X^{vp^h} Y^{(r_h - i_h - v)p^h} \right) \\ &= \prod_{h=0}^k \left(\sum_{j=0}^{r_h} \gamma_{h,j} X^{jp^h} Y^{(r_h-j)p^h} \right) = \sum_{\substack{j=j_0+j_1p+\dots+j_kp^k \\ 0 \leq j_h \leq r_h}} \delta_j X^j Y^{n-j} \in W_{r_0, \dots, r_k}. \end{aligned}$$

Therefore, W_{r_0, r_1, \dots, r_k} is invariant by the action of G and hence it is a KG -module. Let ρ denote the K -representation of G given by W_{r_0, r_1, \dots, r_k} and let χ be the Brauer character associated to the p -modular character afforded by ρ . If $\epsilon = 1$ then $\overline{\zeta_{2m}}$ belongs to the field with q elements, so that the diagonal matrix $D = \text{diag}(\overline{\zeta_{2m}}, \overline{\zeta_{2m}}^{-1})$ belongs to $\text{SL}(2, q)$. After a suitable election of ζ_{2m} we may assume that $g_0 = D$ because g_0 is conjugate to a power of D in G . Then each base element $X^s Y^{n-s}$ is an eigenvector of $\rho(g_0)$ with eigenvalue $\overline{\zeta_{2m}}^{2i-n} = \overline{\zeta_{2m}}^s = \overline{\zeta_m}^{\frac{s}{2}}$ for $s = s_0 + s_1p + \dots + s_kp^k$, $-r_j \leq s_j \leq r_j$ and $2 \mid r_j - s_j$ for each j . Therefore $\chi_{r_0, r_1, \dots, r_k}$ coincides with the restriction of χ to $\langle g_0 \rangle$. Suppose that $\epsilon = -1$. Then $D \in \text{SL}(2, q^2)$ and this group acts on W_{r_0, r_1, \dots, r_k} in the same way. Again we may assume that $g_0 = D$ and the same argument shows that the restriction of χ to $\langle g_0 \rangle$ coincides with $\chi_{r_0, r_1, \dots, r_k}$.

(3) The absolutely irreducible characters in characteristic p of $\text{SL}(2, q)$ have been described in [BN41] (see also [Sri64]). After lifting these characters to G we obtain that if $0 \leq r_0, r_1, \dots, r_{f-1} \leq p-1$ and $r_0 + r_1 + \dots + r_{f-1}$ is even then $\chi_{r_0, r_1, \dots, r_{f-1}}$ is the restriction to $\langle g_0 \rangle$ of an irreducible Brauer character of G modulo p and, conversely, the restriction to $\langle g_0 \rangle$ of any irreducible Brauer character of G modulo p is of this form.

(4) Straightforward. □

Convention: In the remainder of the paper we will often encounter some fixed element $g_0 \in G$. Then we will use the functions $\phi_h, \psi_h, \chi_{r_0, r_1, \dots, r_k} : \langle g_0 \rangle \rightarrow \mathbb{C}$ and we will abuse the notation by referring to the first two as ordinary characters of G and to the last ones as the Brauer characters of G module p , rather as the restriction to $\langle g_0 \rangle$ of such an ordinary or Brauer character. This will be harmless because we will use only these restrictions.

Proposition 4.2. *Let $G = \text{PSL}(2, q)$ with q an odd prime power and let r be a prime not dividing q . Then $\text{TPA}_r(G) = \text{VPA}_r(G)$.*

Proof. The result is trivial if $q \not\equiv \pm 1 \pmod{2r}$ because in such case r does not divide $|G|$ and hence $\text{VPA}_r(G) = \emptyset = \text{TPA}_r(G)$, by Proposition 3.3. The result is also trivial if $r = 2$ because all the elements of G of order 2 are conjugate. So suppose that r is odd and $q \equiv \pm 1 \pmod{2r}$.

Let $\Upsilon \in \text{VPA}_r(G)$. To prove the lemma we fix an element $g_0 \in G$ of order r and use (V4) with the Brauer character χ_2 . We have that $\{g_0^i : i = 1, \dots, \frac{r-1}{2}\}$ is a set of representatives of the conjugacy classes of G of elements of order r . Then, by (V1), (V2) and (V3), we have $\sum_{j=1}^{\frac{r-1}{2}} \Upsilon_1(g_0^j) = 1$. Moreover, for each $l, i \in \{1, \dots, \frac{r-1}{2}\}$ we have

$$\text{Tr}_{\mathbb{Q}(\zeta_r)/\mathbb{Q}}(\chi_2(g_0^i)\zeta_r^{-l}) = \begin{cases} r-3, & \text{if } i \equiv \pm l \pmod{r}; \\ -3, & \text{otherwise.} \end{cases}$$

Hence

$$\mu(\zeta_r^l, \Upsilon, \chi_2) = \frac{1}{r} \left((r-3)\Upsilon_1(g_0^l) - 3 \sum_{j=1, j \neq l}^{\frac{r-1}{2}} \Upsilon_1(g_0^j) + 3 \right) = \Upsilon_1(g_0^l) \in \mathbb{Z}_{\geq 0}.$$

Therefore, there is an integer i in the interval $[1, \frac{r-1}{2}]$ such that $\Upsilon_1(g_0^i) = 1$ and $\Upsilon_1(h) = 0$ for every $h \in G \setminus (g_0^i)^G$. Then Υ is the distribution of partial augmentations of g_0^i . We conclude that $\Upsilon \in \text{TPA}_r(G)$. \square

Corollary 4.3. *Let $G = \text{PSL}(2, q)$ with q an odd prime power and let m be a square-free positive integer. Assume that $q \equiv \pm 1 \pmod{2m}$ and let $\Upsilon \in \text{VPA}_m(G)$. Then G has an element g_0 of order m such that for every prime divisor t of m we have*

$$(8) \quad \Upsilon_{\frac{m}{t}}(g) = \begin{cases} 1, & \text{if } g \in (g_0^{\frac{m}{t}})^G; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Fix an element g_1 of G of order m . By (5) and Proposition 4.2, if t is a prime divisor of m then $\Upsilon_{\frac{m}{t}} \in \text{VPA}_t(G) = \text{TPA}_t(G)$. As every element of order t in G is conjugate in G to an element of $\langle g_1^{\frac{m}{t}} \rangle$, we deduce that there is an integer i_t coprime with t such that $\Upsilon_{\frac{m}{t}}$ is the distribution of partial augmentations of $g_1^{\frac{m}{t}i_t}$. Let i be an integer with $i \equiv \pm i_t \pmod{t}$ for every prime t dividing m and let $g_0 = g_1^i$. Then $g_0^{\frac{m}{t}} = g_1^{\frac{m}{t}i_t}$ for every prime t and hence $\Upsilon_{\frac{m}{t}}$ is the distribution of partial augmentations of $g_0^{\frac{m}{t}}$. In particular, (8) holds for every prime t . \square

In the remainder of the section r and t are different primes such that $q \equiv \pm 1 \pmod{2rt}$ and $\Upsilon \in \text{VPA}_{rt}(G)$. By Corollary 4.3, G has an element g_0 of order rt , which will be fixed for the remainder of the section, such that

$$(9) \quad \Upsilon_r(g) = \begin{cases} 1, & \text{if } g \in (g_0^r)^G; \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad \Upsilon_t(g) = \begin{cases} 1, & \text{if } g \in (g_0^t)^G; \\ 0, & \text{otherwise.} \end{cases}$$

As every element of G of order divisible by rt is conjugate to an element of $\langle g_0 \rangle$, we have that $\Upsilon_1(g) = 0$ for every $g \in G$ which is not conjugate to an element of $\langle g_0 \rangle \setminus \{1\}$. This will simplify the expression in (V4), as in the following lemma.

For a general finite group G , an element $\Upsilon = (\Upsilon_d)_{d|n}$ of $\text{VPA}_n(G)$ and a positive integer m we define the *accumulated virtual partial augmentations* of Υ at m as follows:

$$\tilde{\Upsilon}_d(m) = \sum_{g^G, |g|=m} \Upsilon_d(g).$$

Lemma 4.4. *Let χ be either an ordinary character or a Brauer character of G modulo p . Then*

$$\begin{aligned} \mu(1, \Upsilon, \chi) = & \frac{1}{rt} \left[\tilde{\Upsilon}_1(rt) \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\chi(g_0)) + \tilde{\Upsilon}_1(t) \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\chi(g_0^r)) + \tilde{\Upsilon}_1(r) \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\chi(g_0^t)) + \right. \\ & \left. \text{Tr}_{\mathbb{Q}(\zeta_r)/\mathbb{Q}}(\chi(g_0^t)) + \text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\chi(g_0^r)) + \chi(1) \right] \end{aligned}$$

and

$$0 \leq \mu(1, \Upsilon, \chi) \leq \frac{1}{t} [\chi(1) + \text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\chi(g_0^r))].$$

Proof. If x and y are elements of G with the same order, then x is conjugate in G to a power of y . This implies that if e is a multiple of this common order then there exists $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_e)/\mathbb{Q})$ such that $\chi(x) = \sigma(\chi(y))$. (If χ is a Brauer character modulo p then one assume that $x, y \in G_p$.) Thus $\text{Tr}_{\mathbb{Q}(\zeta_e)/\mathbb{Q}}(\chi(x)) = \text{Tr}_{\mathbb{Q}(\zeta_e)/\mathbb{Q}}(\chi(y))$. Combining this with (V4) we obtain the expression for $\mu(1, \Upsilon, \chi)$ in the lemma. Moreover, $\mu_r^-(\Upsilon, \chi) = \frac{1}{rt}(\chi(1) + \text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\chi(g_0^r)))$, by (9). The inequality is then a consequence of (6) for $n = rt$ and $m = r$. \square

The specialization of the following lemma to the case where Υ is the distribution of partial augmentations of a torsion element of $V(\mathbb{Z}G)$ is a particular case of a result of Wagner [Wag95, BM15].

Lemma 4.5. *t divides $\tilde{\Upsilon}_1(t)$ and r divides $\tilde{\Upsilon}_1(r)$.*

Proof. By symmetry we only have to prove $t \mid \tilde{\Upsilon}_1(t)$. We will give one proof for the case when t is odd and another one for the case when r is odd. This cover all the cases because r and t are different primes. Write $q = \epsilon + 2rth$ with $\epsilon = \pm 1$ and h an integer. We will use Lemma 4.4 and (V4) with the ordinary character ϕ_t (relative to the element g_0 of order $m = rt$ fixed above).

Using Lemma 3.1, for any j we have:

$$\text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\phi_t(g_0^j)) = \begin{cases} 2\epsilon(r-1)(t-1), & \text{if } r \mid j; \\ 2\epsilon(1-t), & \text{if } r \nmid j. \end{cases}$$

Clearly, we also have $\text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\phi_t(g_0^j)) = 2\epsilon(t-1)$ if $\gcd(rt, j) = r$, and $\text{Tr}_{\mathbb{Q}(\zeta_r)/\mathbb{Q}}(\phi_t(g_0^j)) = -2\epsilon$ if $\gcd(rt, j) = t$. By (V1) we have $\tilde{\Upsilon}_1(r) + \tilde{\Upsilon}_1(t) + \tilde{\Upsilon}_1(rt) = 1$. Combining this with Lemma 4.4 we get

$$rt\mu(1, \Upsilon, \phi_t) = 2r\epsilon(t-1)\tilde{\Upsilon}_1(t) + 2rth.$$

Thus, if t is odd then t divides $\tilde{\Upsilon}_1(t)$.

Suppose r odd. Let $1 \leq j \leq \frac{r-1}{2}$ be an integer. We have for every integer i that

$$(10) \quad \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\phi_t(g_0^i)\zeta_{rt}^{-jt}) = \begin{cases} \epsilon(r-2)(t-1), & \text{if } i \equiv \pm j \pmod{r}; \\ 2\epsilon(1-t), & \text{if } i \not\equiv \pm j \pmod{r}. \end{cases}$$

For every $1 \leq i \leq \frac{r-1}{2}$ we denote $x_i = \sum_{1 \leq k \leq \frac{rt}{2}, k \equiv \pm i \pmod{r}} \Upsilon_1(g_0^k)$. Then $\tilde{\Upsilon}_1(t) + \sum_{i=1}^{\frac{r-1}{2}} x_i = 1$. By (10), we obtain that $\sum_{1 \leq k \leq \frac{rt}{2}} \Upsilon_1(g_0^k) \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\phi_t(g_0^k)\zeta_{rt}^{-jt}) = \epsilon(t-1)(-2\tilde{\Upsilon}_1(t) + (r-2)x_j - 2\sum_{i \neq j} x_i) = \epsilon(t-1)(rx_j - 2)$. Moreover, we have $\text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\phi_t(g_0^r)\zeta_{rt}^{-jtr}) = 2\epsilon(t-1)$ and

$$\text{Tr}_{\mathbb{Q}(\zeta_r)/\mathbb{Q}}(\phi_t(g_0^t)\zeta_{rt}^{-jt^2}) = \epsilon w_j \quad \text{with} \quad w_j = \begin{cases} (r-2), & \text{if } jt \equiv \pm 1 \pmod{r}; \\ -2, & \text{if } jt \not\equiv \pm 1 \pmod{r}. \end{cases}$$

Applying condition (V4) with the character ϕ_t and $l = jt$ we deduce that

$$\epsilon(t-1)(rx_j - 2) + 2\epsilon(t-1) + \epsilon w_j + 2rth + 2\epsilon = 2hrt + \epsilon(rx_j(t-1) + 2 + w_j)$$

is a multiple of rt . In particular, t divides $2 + w_j - rx_j$ for every $j = 1, \dots, \frac{r-1}{2}$. Summing for $j = 1, \dots, \frac{r-1}{2}$ and taking into account that $\sum_{j=1}^{\frac{r-1}{2}} w_j = 1$ we obtain that t divides $r \left(1 - \sum_{j=1}^{\frac{r-1}{2}} x_j\right) = r\tilde{\Upsilon}_1(t)$. Therefore $t \mid \tilde{\Upsilon}_1(t)$, as desired. \square

Proposition 4.6. *Let Υ be an element of $\text{VPA}_{rt}(G)$, where $G = \text{PSL}(2, q)$ for an odd prime power q and r and t are different primes with $(rt, q) = 1$. Then*

$$\tilde{\Upsilon}_1(r) = \tilde{\Upsilon}_1(t) = 0 \quad \text{and} \quad \tilde{\Upsilon}_1(rt) = 1.$$

Proof. By symmetry we may assume that $r < t$. We first prove that $\tilde{\Upsilon}_1(t) = 0$. For that we use Lemma 4.4 applied to the Brauer character χ_{2r} . Using Lemma 3.1 we have

$$\begin{aligned} \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\chi_{2r}(g_0)) &= (t-1)(r-1), \\ \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\chi_{2r}(g_0^t)) &= (t-1)(r-1), \quad \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\chi_{2r}(g_0^r)) = (r-1)(t-1-2r), \\ \text{Tr}_{\mathbb{Q}(\zeta_r)/\mathbb{Q}}(\chi_{2r}(g_0^t)) &= r-1 \quad \text{and} \quad \text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\chi_{2r}(g_0^r)) = t-1-2r. \end{aligned}$$

Combining these equalities with $\tilde{\Upsilon}_1(r) + \tilde{\Upsilon}_1(t) + \tilde{\Upsilon}_1(rt) = 1$, we obtain by straightforward calculations that $\mu(1, \Upsilon, \chi_{2r}) = 1 - 2\frac{\tilde{\Upsilon}_1(t)}{t}$ and $\chi_{2r}(1) + \text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\chi_{2r}(g_0^r)) = t$. Using Lemma 4.4, we conclude that $0 \leq \frac{\tilde{\Upsilon}_1(t)}{t} \leq \frac{1}{2}$. By Lemma 4.5, we know that $\frac{\tilde{\Upsilon}_1(t)}{t}$ is an integer, hence $\tilde{\Upsilon}_1(t) = 0$ as desired.

Then $\tilde{\Upsilon}_1(r) + \tilde{\Upsilon}_1(rt) = 1$ and it remains only to show that $\tilde{\Upsilon}_1(r) = 0$. For that we use Lemma 4.4 with the Brauer character χ_2 . In this case we have

$$\begin{aligned} \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\chi_2(g_0)) &= tr - r - t + 3, \\ \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\chi_2(g_0^t)) &= tr - r - 3t + 3, \quad \text{Tr}_{\mathbb{Q}(\zeta_{rt})/\mathbb{Q}}(\chi_2(g_0^r)) = tr - 3r - t + 3, \\ \text{Tr}_{\mathbb{Q}(\zeta_r)/\mathbb{Q}}(\chi_2(g_0^t)) &= r - 3 \quad \text{and} \quad \text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\chi_2(g_0^r)) = t - 3. \end{aligned}$$

Therefore, we have that $\mu(1, \Upsilon, \chi_2) = 1 - 2\frac{\tilde{\Upsilon}_1(r)}{r}$ and $\chi_2(1) + \text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}(\chi_2(g_0^r)) = t$. Applying Lemma 4.4 we obtain $0 \leq \frac{\tilde{\Upsilon}_1(r)}{r} \leq \frac{1}{2}$ and the result follows, since, by Lemma 4.5, $\frac{\tilde{\Upsilon}_1(r)}{r}$ is an integer. \square

5. PROOF OF THEOREM 2.3

In this section we prove Theorem 2.3. So t is an odd prime integer, q is a prime power with $q \equiv \pm 1 \pmod{4t}$ and $G = \text{PSL}(2, q)$. If $t = 3$ then G has a unique conjugacy class of elements of order 3 and a unique one of elements of order 6. Combining this with Proposition 4.6 we deduce that $\text{VPA}_6(G) = \text{TPA}_6(G)$. The specialization of this to partial augmentations of torsion units is a result of Hertweck [Her07, Proposition 6.6]. So in the remainder we assume that $t \geq 5$. We set $n = \frac{t-1}{2}$.

We first prove the inclusion $\text{VPA}_{2t}(G) \subseteq \text{TPA}_{2t}(G) \cup \{\Upsilon^{(g_0)} : g_0 \in G, |g_0| = 2t\}$. For that we take an element $\Upsilon = (\Upsilon_1, \Upsilon_2, \Upsilon_t, \Upsilon_{2t})$ of $\text{VPA}_{2t}(G)$ and we show that there is an element g_0 of order $2t$ in G such that either $\Upsilon = \varepsilon^*$ or $\Upsilon = \Upsilon^{(g_0)}$, where ε^* is the distribution of partial augmentations of g_0 and $\Upsilon^{(g_0)}$ is as in (3). By (V3), if g is an element of G of order not dividing $2t$ then $\Upsilon_d(g) = 0$ for each $d \mid 2t$. By the results of Section 4, there exists $g_0 \in G$ of order $2t$ such that Υ_t and Υ_2 are as in (9) with $r = 2$. This shows that $\Upsilon_t = \Upsilon_t^{(g_0)} = \varepsilon_t^*$ and $\Upsilon_2 = \Upsilon_2^{(g_0)} = \varepsilon_2^*$, where ε^* is the distribution of partial augmentations of g_0 . As, clearly $\Upsilon_{2t} = \Upsilon_{2t}^{(g_0)} = \varepsilon_{2t}^*$, it remains to show that Υ_1 is either ε_1^* or $\Upsilon_1^{(g_0)}$.

Let Odd denote the set of odd integers in the interval $[1, t-1]$ and for each $l \in [1, t-1]$ let

$$i_l = \begin{cases} l, & \text{if } l \in \text{Odd}; \\ t-l, & \text{otherwise;} \end{cases} \quad w_l = \begin{cases} 1, & \text{if } l \in \{1, t-1\}; \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad W_l = \begin{cases} 1, & \text{if } l \in \{1, 2, t-2, t-1\}; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that i_l is the unique element $j \in \text{Odd}$ with $j \equiv \pm l \pmod{t}$ and i_{nl} is the unique element $j \in \text{Odd}$ with $2j \equiv \pm l \pmod{t}$. Moreover, $i_l \neq i_{nl}$ because $n \not\equiv \pm 1 \pmod{t}$, as $t \geq 5$.

We also use the notation $T = \text{Tr}_{\mathbb{Q}(\zeta_{2t})/\mathbb{Q}} = \text{Tr}_{\mathbb{Q}(\zeta_t)/\mathbb{Q}}$.

Observe that $\{g_0^i : i \in \text{Odd}\}$ and $\{g_0^{t-i} : i \in \text{Odd}\}$ are sets of representatives of the conjugacy classes of elements of G of orders $2t$ and t , respectively. By Proposition 4.6 we have

$$(11) \quad \Upsilon_1(g_0^t) = \sum_{i \in \text{Odd}} \Upsilon_1(g_0^{t-i}) = 0 \quad \text{and} \quad \sum_{i \in \text{Odd}} \Upsilon_1(g_0^i) = 1.$$

By (V4) and Lemma 4.1.(2) if m is an even integer and $l \in \mathbb{Z}$ then

$$(12) \quad \frac{1}{2t} \left[\chi_m(1) + \chi_m(g_0^t) (-1)^l + T \left(\chi_m(g_0^2) \zeta_{2t}^{-2l} \right) \right. \\ \left. + \sum_{i \in \text{Odd}} \left(\Upsilon_1(g_0^i) T \left(\chi_m(g_0^i) \zeta_{2t}^{-l} \right) + \Upsilon_1(g_0^{t-i}) T \left(\chi_m(g_0^{t-i}) \zeta_{2t}^{-l} \right) \right) \right] \in \mathbb{Z}_{\geq 0}.$$

We will use this with $l \in \{1, \dots, t-1\}$ and $m \in \{2, 4\}$. To facilitate the calculations we collect the following equalities which are all direct application of Lemma 3.1:

$$\chi_2(1) = 3, \quad \chi_2(g_0^t) = -1, \quad \chi_4(1) = 5, \quad \chi_4(g_0^t) = 1, \\ T \left(\chi_2(g_0^2) \zeta_{2t}^{-2l} \right) = tw_l - 3, \quad T \left(\chi_4(g_0^2) \zeta_{2t}^{-2l} \right) = tW_l - 5.$$

Moreover, if $i \in \text{Odd}$ then

$$T \left(\chi_2(g_0^i) \zeta_{2t}^{-l} \right) = \begin{cases} (1-t)(-1)^l, & \text{if } i = i_l; \\ (-1)^l, & \text{otherwise;} \end{cases} \quad T \left(\chi_4(g_0^i) \zeta_{2t}^{-l} \right) = \begin{cases} (t+1)(-1)^{l+1}, & \text{if } i = i_l; \\ (t-1)(-1)^l, & \text{if } i = i_{nl}; \\ (-1)^{l+1}, & \text{otherwise;} \end{cases} \\ T \left(\chi_2(g_0^{t-i}) \zeta_{2t}^{-l} \right) = \begin{cases} (t-3)(-1)^l, & \text{if } i = i_l; \\ (-3)(-1)^l, & \text{otherwise;} \end{cases} \quad T \left(\chi_4(g_0^{t-i}) \zeta_{2t}^{-l} \right) = \begin{cases} (t-5)(-1)^l, & \text{if } i \in \{i_l, i_{nl}\}; \\ (-5)(-1)^l, & \text{otherwise.} \end{cases}$$

Plugging this information in (12) for $m = 2$ and $m = 4$ and using (11), we obtain

$$(13) \quad \frac{1}{2} \left((-1)^l \left(\Upsilon_1(g_0^{t-i_l}) - \Upsilon_1(g_0^{i_l}) \right) + w_l \right) \in \mathbb{Z}_{\geq 0}$$

and

$$(14) \quad \frac{1}{2} \left((-1)^l \left(\Upsilon_1(g_0^{i_{nl}}) + \Upsilon_1(g_0^{t-i_{nl}}) - \Upsilon_1(g_0^{i_l}) + \Upsilon_1(g_0^{t-i_l}) \right) + W_l \right) \in \mathbb{Z}_{\geq 0}.$$

Using (13) with l and with $t-l$ we obtain $|\Upsilon_1(g_0^l) - \Upsilon_1(g_0^{t-l})| \leq w_l$ if $l \in \text{Odd}$. In particular,

$$(15) \quad \Upsilon_1(g_0^l) = \Upsilon_1(g_0^{t-l}), \text{ if } l \in \text{Odd} \setminus \{1\}.$$

This together with (11) yields

$$(16) \quad \Upsilon_1(g_0) = 1 - \sum_{l \in \text{Odd} \setminus \{1\}} \Upsilon_1(g_0^l) = 1 - \sum_{l \in \text{Odd} \setminus \{1\}} \Upsilon_1(g_0^{t-l}) = 1 + \Upsilon_1(g_0^{t-1}).$$

We now combine (15) and (16) with (14) for l and $t-l$. When we take $l = 1$ we obtain

$$\Upsilon_1(g_0^{i_n}) = \Upsilon_1(g_0^{t-i_n}) \in \{0, 1\};$$

and when we take $l \in \text{Odd} \setminus \{1, t-2\}$ we have

$$\Upsilon_1(g_0^{i_{nl}}) = \Upsilon_1(g_0^{t-i_{nl}}) = 0, \quad \text{if } l \in \text{Odd} \setminus \{1, t-2\}.$$

As $l \mapsto i_{nl}$ defines a bijection $\text{Odd} \rightarrow \text{Odd}$ mapping $t-2$ to 1, the latter is equivalent to

$$\Upsilon_1(g_0^l) = \Upsilon_1(g_0^{t-l}) = 0, \quad \text{for all } l \in \text{Odd} \setminus \{i_n, 1\}.$$

Thus

$$\Upsilon_1(g_0) = 1 + \Upsilon_1(g_0^{t-1}) = 1 - \sum_{l \in \text{Odd} \setminus \{1\}} \Upsilon_1(g_0^{t-l}) = 1 - \Upsilon_1(g_0^{t-i_n}) = 1 - \Upsilon_1(g_0^{i_n}) \in \{0, 1\}.$$

Since $\{i_n, t-i_n\} = \{n, n+1\}$, we conclude that either $\Upsilon_1(g_0) = 1$ and $\Upsilon_1(g_0^l) = 0$ for every $2 \leq l \leq t-1$, or $\Upsilon_1(g_0^{t-1}) = -1$, $\Upsilon_1(g_0^n) = \Upsilon_1(g_0^{n+1}) = 1$ and $\Upsilon_1(g_0^l) = 0$ for every integer l in $[1, t-1] \setminus \{n, n+1, t-1\}$. In the first case $\Upsilon_1 = \varepsilon_1^*$ and in the latter case $\Upsilon_1 = \Upsilon_1^{(g_0)}$, as desired. This finishes the necessary part of the proof.

To finish the proof of Theorem 2.3 it remains to prove that if g_0 is an element of G of order $2t$ then $\Upsilon^{(g_0)} \in \text{VPA}_{2t}(G)$. That $\Upsilon^{(g_0)}$ satisfies conditions (V1), (V2) and (V3) follows by straightforward arguments. So it remains to show that the following is a non-negative integer for every ordinary or Brauer character:

$$\mu(\zeta_{2t}^l, \Upsilon^{(g_0)}, \chi) = \frac{1}{2t} \left[\chi(1) + \chi(g_0^t) (-1)^l + T \left(\chi(g_0^2) \zeta_{2t}^{-2l} + (\chi(g_0^n) + \chi(g_0^{n+1}) - \chi(g_0^{t-1})) \zeta_{2t}^{-l} \right) \right].$$

Actually, it suffices to consider Brauer characters modulo the prime p dividing q . This is a consequence of the following remark, which was brought to our attention by Leo Margolis.

Remark 5.1. *Let $G = \text{PSL}(2, q)$, with q a power of a prime p and let n be an integer coprime to p . Let $\Upsilon = (\Upsilon_d)_{d|n}$ be a list of class functions of G satisfying conditions (V1), (V2) and (V3). Then $\Upsilon \in \text{VPA}_n(G)$ if and only if it satisfies (V4) for every irreducible Brauer character of G modulo p .*

Proof. Suppose that $\Upsilon \in \text{VPA}_n(G)$ satisfies (V4) for the irreducible Brauer characters of G modulo p . Observe that $\mu(\zeta_n^l, \Upsilon, \chi)$ is \mathbb{Z} -linear in the last argument. This implies that if condition (V4) holds for the class functions f_1, \dots, f_k then it also holds for each linear combination $\chi = a_1 f_1 + \dots + a_k f_k$ with a_1, \dots, a_k non-negative integers. This is also valid for class functions defined on the t -regular elements of G for a given prime t . Therefore, to verify (V4) it is enough to consider irreducible ordinary characters and irreducible Brauer characters of G . Moreover, as n is coprime with p , by (V3), each non-zero summand in the expression of $\mu(\zeta_n^l, \Upsilon, \chi)$ correspond to p -regular elements, for any ordinary or Brauer character of G . Thus we only have to consider the restriction of each ordinary character of G to the p -regular elements and the restriction of Brauer characters of G modulo a prime t to the $\{p, t\}$ -regular elements. If t is a prime integer then the restriction to the t -regular elements of G of an irreducible ordinary character of G is a linear combination of the irreducible Brauer characters of G modulo t with coefficients in the decomposition matrix relative to t and these coefficients are non-negative (see e.g. [Ser78, Section 15.2]). Using the expression of the ordinary characters of G on the p -regular elements in terms of the Brauer characters of G modulo p we deduce that (V4) holds for all the ordinary characters of G . Let now t be a prime different from p . The decomposition matrix A of G relative to t is described in [Bur76]. Every non-zero column of A contains an entry equal to 1 in a row on which all the other entries are 0. This implies that each Brauer character of G modulo t equals the restriction to the t -regular elements of an ordinary character of G . Hence (V4) also holds for Brauer characters of G modulo t . \square

By Lemma 4.1.(3) and Remark 5.1, it remains to show that if r_0, r_1, \dots, r_k are non-negative integers with $r_0 + \dots + r_k$ even and l is an integer then $\mu(\zeta_{2t}^l, \Upsilon^{(g_0)}, \chi_{r_0, r_1, \dots, r_k}) \in \mathbb{Z}_{\geq 0}$. For that we use that the map $\chi \mapsto \mu(\zeta_{2t}^l, \Upsilon^{(g_0)}, \chi)$ is linear and the expression of $\chi_{r_0, r_1, \dots, r_k}$ obtained in Lemma 4.1.(4) in terms of the ordinary characters 1_G , ϕ_h and ψ_h for $h \in \{1, \dots, t\}$. Hence we start considering the latter characters.

In the remainder of the proof l is an integer, $h \in \{1, \dots, t\}$ and $\epsilon = \pm 1$ with $q \equiv \epsilon \pmod{4t}$. We will use Lemma 3.1 without specific mention. An easy calculation shows that

$$(17) \quad \mu(\zeta_{2t}^l, \Upsilon^{(g_0)}, \psi_h) = \frac{q - \epsilon}{2t} \quad \text{and} \quad \mu(\zeta_{2t}^l, \Upsilon^{(g_0)}, 1_G) = \begin{cases} 1, & \text{if } 2t \mid l; \\ 0, & \text{otherwise.} \end{cases}$$

Now we calculate $\mu(\zeta_{2t}^l, \Upsilon^{(g_0)}, \phi_h)$. For that we introduce the following notation:

$$\vartheta_{h,l} = \begin{cases} 1, & \text{if } h \equiv \pm 2l \pmod{2t}, \text{ or } h \equiv \pm l \pmod{t} \text{ and } 2 \nmid l; \\ 0, & \text{otherwise;} \end{cases} \quad \gamma_{i,l} = \begin{cases} t, & \text{if } hi \equiv \pm l \pmod{t}; \\ 0, & \text{if } hi \not\equiv \pm l \pmod{t}; \end{cases}$$

for each $1 \leq i \leq t-1$. Clearly, we have $\phi_h(g_0^t) = 2\epsilon(-1)^h$, $T(\phi_h(g_0^i)\zeta_{2t}^{-l}) = (\gamma_{i,l} - 2)\epsilon(-1)^{hi+l}$, $\gamma_{n,l} = \gamma_{n+1,l}$, $\gamma_{2,2l} = \gamma_{t-1,l} = \gamma_{1,l}$ and

$$\gamma_{1,l}(1 - (-1)^l) + \gamma_{n,l}(-1)^{hn+l}(1 + (-1)^h) = 2t\vartheta_{h,l}.$$

Therefore

$$T\left(\phi_h(g_0^2)\zeta_{2t}^{-2l} - \phi_h(g_0^{t-1})\zeta_{2t}^{-l}\right) = \epsilon(\gamma_{1,l} - 2)(1 - (-1)^l)$$

and

$$T\left((\phi_h(g_0^n) + \phi_h(g_0^{n+1}))\zeta_{2t}^{-l}\right) = \epsilon(\gamma_{n,l} - 2)(-1)^{hn+l}(1 + (-1)^h) = \begin{cases} 2\epsilon(\gamma_{n,l} - 2)(-1)^l, & \text{if } 2 \mid h; \\ 0, & \text{if } 2 \nmid h. \end{cases}$$

Hence

$$(18) \quad \mu\left(\zeta_{2t}^l, \Upsilon^{(g_0)}, \phi_h\right) = \frac{q - \epsilon + \epsilon(\gamma_{1,l}(1 - (-1)^l) + \gamma_{n,l}(-1)^{hn+l}(1 + (-1)^h))}{2t} = \frac{q - \epsilon}{2t} + \epsilon\vartheta_{h,l}.$$

Let $R = (r_0, r_1, \dots, r_k) \in \mathbb{Z}^{k+1}$ with $r_0 + r_1 + \dots + r_k$ even, and let $2n_h$ be the cardinality of the set $V_{R,h}$ defined in (7). Observe that $\mu(\xi, \Upsilon, \chi)$ is linear in the third argument. Then, using Lemma 4.1.(4), (17) and (18) we obtain

$$(19) \quad \mu(\zeta_{2t}^l, \Upsilon^{(g_0)}, \chi_{r_0, \dots, r_k}) = k_0\mu(\zeta_{2t}^l, \Upsilon^{(g_0)}, 1_G) + \sum_{h=1}^t n_h\vartheta_{h,l},$$

where k_0 is either $2n_0$ or $1 + 2n_0$. This finishes the proof of Theorem 2.3, as the expression in (19) is a non-negative integer because $\mu(\zeta_{2t}^l, \Upsilon^{(g_0)}, 1_G)$ is either 0 or 1 by (17), and all the n_h are non-negative integers.

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